DESIGN AND ANALYSIS OF PARTIALLY ACCELERATED LIFE TESTING PLAN FOR GOMPERTZ DISTRIBUTION WITH PROGRESSIVE FIRST-FAILURE CENSORING

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ABSTRACT

The study is to establish a progressive first-failure censoring step-stress partially accelerated life testing plan for Gompertz distribution. The distribution parameters and acceleration factor are estimated using maximum likelihood estimation (MLEs). Additionally, the estimators' asymptotic variance, confidence intervals, and covariance matrices are provided. To obtain the optimal stress change time for the step-stress partially accelerated life test, the asymptotic variance of the MLEs of the parameters is minimized. The results of the simulations are used to determine the precision of the MLEs for the parameters under consideration.

Keywords: Life testing & reliability; Gompertz distribution; Progressive first-failure censoring; Optimal stress change time; Simulation study.

INTRODUCTION

Manufacturers today are under intense pressure to develop more advanced, sophisticated technology items quickly while also increasing production. This has prompted the development of techniques like concurrent engineering and an increase in the use of designed experiments to improve products and processes. Increased reliability standards have necessitated additional upfront testing of products and systems. This is consistent with the modern quality concept for generating high-reliability products: increase reliability through improved design and manufacturing processes; eliminate dependency on inspection (or screening) to achieve high reliability.

For decades, quality engineers have employed accelerated life test (ALT) procedures to rapidly gather reliable data in manufacturing firms (Hakamipour & Rezaei, 2017). ALT's stand as a tool to estimate the lifetime of highly dependable items within the constraints of a suitable testing period. The items are stressed beyond their normal operating limits in order to trigger early breakdowns. The data from the accelerated test is analyzed using a model and then approximated to the design stress in order to evaluate the life distribution (Ahmad et al., 2006). Please see Nelson (1990) for additional details. The recent developments can be seen in Sabri-Laghaie and Noorossana (2021).

Life testing is accelerated through the use of SSALT (SSALT), which is an advanced version of ALT. To begin implementing the SSALT, we begin by applying minimal stress to all products. If the product withstands the stress (i.e. does not fail), we increase the stress level; if only one change in the stress level is made, this is referred to as a basic step-stress test. When

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many changes in the stress level are made, this is referred to as a multiple step-stress test. Numerous writers have recently explored the SSALT and its design optimization: (Wu et al., 2008), Balakrishnan and Han (2009), Bakoban (2012), Han (2017).When a test consists of two levels of stress, the first of which is normal, and the second of which is changed at a predetermined time point, it is said to be a step-stress partially ALT (SSPALT). For example, see Goel (1971), DeGroot and Goel (1979). In a step-stress PALT under censored data, (Abdel-Ghaly et al., 2002) studied the MLE approach for parameters of the Weibull distribution in an effort to improve accuracy. (Mohie El-Din et al., 2015) estimated the parameters of Weibull Distribution under SSPALT with Progressive First-Failure Censoring. (Lone et al., 2017) described the adaptive type-I progressive hybrid censoring to develop a step stress partially accelerated life testing plan for competing risk. (Rahman et al., 2018) used progressive type-II censoring to obtain the likelihood estimation of the exponentiated exponential distribution under a step stress partially accelerated life testing plan. Lone and Ahmed (2021) published a complete study and design of Accelerated Life Testing with a Rebate Warranty Application. Recently, (Ahmed et al., 2020) described inferring the burr type X distribution using geometric processes in an accelerated life testing design with time-censored data. More recently, Lone (2021) published a work on multiply censored partially accelerated life testing for the Power Function Model.

In ALTs or PALTs, testing are frequently terminated prior to the failure of all units. Estimates based on censored data are less precise than those based on complete data. This is more than compensated for by the shortened test time and expense. The most often used censoring scheme is type-II censoring. If the experimenter subjects n units to a life test, the experiment is terminated when a predetermined number of units $m < n$ fails. Only the shortest lives are observed in this circumstance. Conventional type-II censoring techniques prohibit the removal of units at any point other than the experiment's terminal point. Progressive type-II censoring is a technique that permits efficient resource utilization by continuously removing a predetermined number of test units that remain functional over the test duration, see, Balakrishnan and Aggarwala (2000). On the other side, units may be removed prior to failure intentionally to save time and money, or when certain materials must be removed to be used in another experiment. According to Balasooriya (1995), in a circumstance when a product's lifetime is rather long and test facilities are rare but test material is reasonably inexpensive, one can test $k \times n$ units by testing n sets, each comprising k units. The life test is then conducted independently on each of these sets of units until the first failure occurs in each set. This type of censoring is referred to as first-failure censoring. Notably, a first-failure censoring technique is stopped upon observation of the first failure in each sample. If an experimenter wishes to eliminate some units prior to detecting the first failures in these sets, the approach outlined above will be ineffective. The first failure censoring prevents sets from being removed from the test prior to the final termination point. This tolerance, however, will be useful surviving items in the test can be employed for future testing. As with unintentional unit breaking or loss of touch with subjects, the loss of test units at places other than the termination point may be inevitable. It enables the removal of a portion of the surviving units from the test at each failure moment and hence can be very effective in reducing the testing time and expenses. This method of censoring will be discussed in further detail in the following section.

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The paper is structured as follows: Section 2 describes the progressive first-failure censoring mechanism, a description of the model, the test technique, and the model's assumptions. Section 3 discovers the MLEs for the SSPALT model parameters and constructs asymptotic confidence bounds for the model parameters using the asymptotic distribution of MLEs. Section 4 contains the findings of the simulation. Section 5 provides the conclusion.

MODEL AND TEST METHOD

A Progressive First-Failure-Censoring Scheme

This section describes how to combine first-failure with progressive censoring, as in Wu and Kus (2009). Assume that n independent groups comprised of k items each are subjected to a life test. The W_1 groups and the group in which the first failure occurs are withdrawn from the test at random following the occurrence of the first failure (say $T^W_{1:m:n:k}$). After the second failure (say $T^W_{2:m:n:k}$) occurs, the W_2 groups and the group in which the second failure occurs are randomly withdrawn from the test. Finally, the W_m groups and the group in which the m^{th} failure occurs are randomly withdrawn from the test as soon as the m^{th} failure occurs (say $T^W_{m:m:n:k}$).Then $T^W_{1:m:n:k} < T^W_{2:m:n:k} < \cdots < T^W_{m:m:n:k}$ are called progressively first-failure-censored order statistics with the progressive censoring scheme W. It is clear that $n = m + W_1 + W_2 + \cdots + W_m$. Assuming that the failure times of the $n \times k$ items primarily used in the test are drawn from a continuous population with a distribution function $F(t)$ and a probability density function $f(t)$, the joint probability density function for $T^W_{1:m:n:k} < T^W_{2:m:n:k} < \cdots < T^W_{m:m:n:k}$ can be calculated as follows.

$$f_{1,2,\dots,m}(T^W_{1:m:n:k} < T^W_{2:m:n:k} < \cdots < T^W_{m:m:n:k}) = dk^m \prod_{i=1}^{m} f(T^W_{i:m:n:k}) \{1 - F(T^W_{i:m:n:k})\}^{k(W_i+1)-1}, \tag{1}$$

$$0 < T^W_{1:m:n:k} < T^W_{2:m:n:k} < \cdots < T^W_{m:m:n:k} < \infty$$

Where

$$d = n(n - W_1 - 1)(n - W_1 - W_2 - 2) \dots (n - W_1 - W_2 - \cdots - W_{m-1} - m + 1). \tag{2}$$

This censoring scheme has the advantage of reducing test time by allowing for the use of more items while ensuring that only m of $n \times k$ items are failures. Notably, certain censoring rules can be accommodated using the preceding notation, including the first-failure censored order statistics when $W = (0, 0, \dots, 0)$, the progressive type-II censored order statistics when $k = 1$, the usual type-I censored order statistics when $k = 1$ and $W = (0, 0, \dots, n - m)$, and a complete sample case when $k = 1$ and $W = (0, 0, \dots, 0)$, with $n = m$. Furthermore, the progressive first-failure censored sample $T^W_{1:m:n:k} < T^W_{2:m:n:k} < \cdots < T^W_{m:m:n:k}$ with distribution function $F(t)$ can be considered as a progressive type-II censored sample from a population with distribution function $1 - \left(1 - F(t)\right)^k$.

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Model Description

A number of authors have made significant contributions to the statistical methodology and characterization of the GD, which was first introduced by Gompertz (1825). For example, (Garg et al.. 1970), Read (1983), Gordon (1990), Makany (1991), Franses (1994), Wu and Lee (1999). The parameters of the GD under type II censoring were studied by Chen (1997). To produce the Gompertz distribution's parameters under the first-failure censored sampling plan, (Wu et al., 2003) devised an exact confidence interval and an exact joint confidence region. In recent times, much attention has been paid to statistical inference for the Gompertz distribution. (Soliman et al., 2012), used the progressive first-failure censored data to estimate the parameters of Gompertz model as a life testing distribution Wu and Shi (2016), considered the Bayes estimations of the Gompertz distribution in a competing risks model with progressively hybrid censoring. (Liu et al., 2018) investigated the Gompertz distribution for estimating the stress-strength reliability of a system with multiple component types. (Mohie El-Din et al., 2019) examined the Gompertz distribution's MLEs and BEs using a generalized progressively hybrid censored scheme.

The probability density function (pdf) of GD is given by:

$$f(x;\alpha,\beta) = \beta exp\left(\alpha t - \frac{\beta}{\alpha}(e^{\alpha t} - 1)\right), \qquad t > 0,\ \alpha,\beta > 0, \tag{3}$$

and the cumulative distribution function (cdf) is given by:

$$F(x;\alpha,\beta) = 1 - exp\left(-\frac{\beta}{\alpha}(e^{\alpha t} - 1)\right), \tag{4}$$

Where α denotes the shape and β denotes the scale parameter. Notably, when α approaches 0, the GD tends towards an exponential distribution. The Gompertz distribution has a unimodal pdf characterized by positive skewness and an increasing hazard rate function given by:

$$H(t) = \beta e^{\alpha t}$$

The Gompertz distribution has been derived as a truncated form of the type-I extreme value distribution; see (Johnson et al,. 1995). Using the gamma distribution as the basis for one of the parameters of the Gompertz distribution, Osman (1987) derived the compound Gompertz model, which is now a widely used lifetime distribution in reliability practice, especially for analyzing survival data in heterogeneous populations.

TEST METHOD

In this subsection, the test method under SSPLAT can be described as follows:

Suppose n identical and independent groups with k items each are subjected to a life test assuming the life of items follows GD. The procedure involves to terminate the test at a time when the prefixed number of failures "m" $(m < n)$ are obtained. Each of the $n \times k$ units is first run under normal operating conditions and then placed under accelerated conditions if it does not

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fail or is not removed from the test within a predetermined time τ. At the i^{th} failure, a random number of the surviving groups $W_i, i = 1, 2, \dots, m-1$, and the group in which the failure $T^W_{i:m:n:k}$ has occurred are randomly selected and removed from the test. Finally, at the m^{th} failure, all the remaining surviving groups $W_m = n - m - \sum_{i=1}^{m-1} W_i$ are removed from the test and the test is terminated.

Let n_1 represent the number of failures that occurred before time τ under normal conditions, and n_2 represent the number of failures that occurred after time τ under stress conditions. Hence, the progressive first-failure censored data observed is as follows:

$$T^W_{1:m:n:k} < \cdots < T^W_{n_1:m:n:k} < \tau < T^W_{n_1+1:m:n:k} < \cdots < T^W_{m:m:n:k},$$

Where

$$W = (W_1, W_2, \dots, W_m) \text{ and} \sum_{i=1}^{m} W_i = m - n.$$

In SS-PALT, the total lifetime of a test item, denoted by T, is divided into two stages: the normal stage and the accelerated stage. As a result, the test unit's lifetime is as follows:

$$T = \begin{cases} Y, & Y \leq \tau \\ \tau + \lambda^{-1}(Y - \tau), & Y > \tau \end{cases},$$

Where Y denotes the product's lifetime under normal usage conditions, τ denotes the time of the stress shift, and λ (>1) denotes the acceleration factor, which is the ratio of the product's mean life under regular use conditions to its mean life under the accelerated situation.

Assume that the lifetime of the test product follows a GD with shape parameter α and scale parameter β. The probability density function and cumulative density function of a product's total lifetime, T, are defined by Equations (5) and (6).

$$F(t;\alpha,\beta,\lambda) = \begin{cases} 0, & t < 0 \\ F_1(t) = 1 - exp\left(-\frac{\beta}{\alpha}\left(e^{\alpha t} - 1\right)\right), & 0 < t \leq \tau \\ F_2(t) = 1 - exp\left[-\frac{\beta}{\alpha}\left(e^{\alpha\{\tau + \lambda(t-\tau)\}} - 1\right)\right], & t > \tau \end{cases} \quad (5)$$

And,

$$f(t;\alpha,\beta,\lambda) = \begin{cases} 0, & t < 0 \\ f_1(t) = \beta exp\left(\alpha t - \frac{\beta}{\alpha}\left(e^{\alpha t} - 1\right)\right), & 0 < t \leq \tau \\ f_2(t) = \beta\lambda exp\left[\alpha\{\tau + \lambda(t-\tau)\} - \frac{\beta}{\alpha}\left(e^{\alpha\{\tau + \lambda(t-\tau)\}} - 1\right)\right], & t > \tau \end{cases} \quad (6)$$

PARAMETERS ESTIMATION

Maximum likelihood (ML) methods are versatile and can be applied to a wide variety of models and data types. They provide efficient techniques for quantifying uncertainty via

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confidence intervals. Although the methodology for estimating the maximum probability is simple, the implementation may require considerable mathematical skill. However, with modern computational techniques, mathematical complexity is not a significant barrier.

The following section describe the procedure of obtaining the point and interval estimations of the parameters $\alpha, \beta,$ and λ based on test method and censoring scheme under consideration.

Point Estimation

Let $t_i = t^W_{i;m,n,k}, i = 1, 2, \dots, m,$ be the observed values of the lifetime T obtained from a progressive first-failure censoring scheme under SSPALT. Then the ML function of the failure times $t_1 < \cdots < t_{n_1} < \tau < t_{n_1+1} < \cdots < t_m$with censoring scheme $W = (W_1, W_2, \dots, W_m)$.

$$L(\alpha,\beta,\lambda) = dk^m \prod_{i=1}^{n_1} f_1(t_i)\{1 - F_1(t_i)\}^{k(W_i+1)-1} \prod_{i=n_1+1}^{m} f_2(t_i)\{1 - F_2(t_i)\}^{k(W_i+1)-1}, \quad (7)$$

where d is defined by (1). Using equation (6), we get

$$L(\alpha,\beta,\lambda) = dk^m \prod_{i=1}^{n_1}[\beta e^{\alpha t_i} exp\{-\beta k\alpha^{-1}(W_i+1)(e^{\alpha t_i}-1)\}] \times \prod_{i=n_1+1}^{m}\left[\beta\lambda e^{\alpha\{\lambda(t_i-\tau)+\tau\}} exp\left\{-\beta k\alpha^{-1}(W_i+1)\left(e^{\alpha\{\lambda(t_i-\tau)+\tau\}}-1\right)\right\}\right].$$

The log-likelihood function is given by:

$$l = l(\alpha,\beta,\lambda) = log(dk^m) + mlog\beta + (m-n_1)log\lambda + \alpha\sum_{i=1}^{n_1} t_i - \beta k\alpha^{-1}\sum_{i=1}^{n_1}\{(W_i+1)(e^{\alpha t_i}-1)\} + \alpha\sum_{i=n_1+1}^{m}\{\lambda(t_i-\tau)+\tau\} - \beta k\alpha^{-1}\sum_{i=n_1+1}^{m}\left[(W_i+1)\left(e^{\alpha\{\lambda(t_i-\tau)+\tau\}}-1\right)\right]. \quad (8)$$

Hence the likelihood equations for parameters $\alpha, \beta,$ and λ are given by:

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$$\frac{\partial l}{\partial \alpha}=\sum_{i=1}^{n_1} t_i-\beta k\left[\alpha^{-1}\sum_{i=1}^{n_1} t_i(W_i+1)e^{\alpha t_i}-\alpha^{-2}\sum_{i=1}^{n_1}\{(W_i+1)(e^{\alpha t_i}-1)\}\right]$$
$$+\sum_{i=n_1+1}^{m}\{\lambda(t_i-\tau)+\tau\}$$
$$-\beta k\left[\alpha^{-1}\sum_{i=n_1+1}^{m}\{\lambda(t_i-\tau)+\tau\}(W_i+1)e^{\alpha\{\lambda(t_i-\tau)+\tau\}}\right.$$
$$\left.-\alpha^{-2}\sum_{i=n_1+1}^{m}\left\{(W_i+1)\left(e^{\alpha\{\lambda(t_i-\tau)+\tau\}}-1\right)\right\}\right]=0 \quad (9)$$

$$\frac{\partial l}{\partial \beta}=m\beta^{-1}-k\alpha^{-1}\sum_{i=1}^{n_1}\{(W_i+1)(e^{\alpha t_i}-1)\}$$
$$-k\alpha^{-1}\sum_{i=n_1+1}^{m}\left[(W_i+1)\left(e^{\alpha\{\lambda(t_i-\tau)+\tau\}}-1\right)\right]=0 \quad (10)$$

$$\frac{\partial l}{\partial \lambda}=\frac{(m-n_1)}{\lambda}+\alpha\sum_{i=n_1+1}^{m}(t_i-\tau)-\beta k\lambda\sum_{i=n_1+1}^{m}\left[(W_i+1)(t_i-\tau)\left(e^{\alpha\{\lambda(t_i-\tau)+\tau\}}\right)\right]=0 \quad (11)$$

The above nonlinear system of equations contains three unknowns, α, β and λ. It is obvious that obtaining a closed-form solution is extremely difficult. As a result, a numerical solution to the above nonlinear system can be found using an iterative procedure such as Newton Raphson.

Interval Estimation

This sub-section is to investigate approximate confidence intervals for the GD parameters $\Theta=(\alpha,\beta,\lambda)$ using progressive first-failure censoring.

From the log-likelihood function in (8), we have

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$$\frac{\partial^2 l}{\partial \alpha^2} = \beta k \Bigg[-\alpha^{-1} \left\{ \sum_{i=1}^{n_1} t_i^2 (W_i + 1) e^{\alpha t_i} + \sum_{i=n_1+1}^{m} \{\lambda(t_i - \tau) + \tau\}^2 (W_i + 1) e^{\alpha\{\lambda(t_i-\tau)+\tau\}} \right\}$$
$$+ 2\alpha^{-2} \left\{ \sum_{i=1}^{n_1} t_i (W_i + 1) e^{\alpha t_i} + \sum_{i=n_1+1}^{m} \{\lambda(t_i - \tau) + \tau\} (W_i + 1) e^{\alpha\{\lambda(t_i-\tau)+\tau\}} \right\}$$
$$- 2\alpha^{-3} \Bigg\{ \sum_{i=1}^{n_1} [(W_i + 1)(e^{\alpha t_i} - 1)]$$
$$+ \sum_{i=n_1+1}^{m} \left[(W_i + 1)\left(e^{\alpha\{\lambda(t_i-\tau)+\tau\}} - 1\right) \right] \Bigg\} \Bigg], \quad (12)$$

$$\frac{\partial^2 l}{\partial \beta^2} = -\frac{m}{\beta^2}, \quad (13)$$

$$\frac{\partial^2 l}{\partial \lambda^2} = \frac{(n_1 - m)}{\lambda} - \beta k \left[\sum_{i=n_1+1}^{m} \left\{ (t_i - \tau)(W_i + 1) e^{\alpha\{\lambda(t_i-\tau)+\tau\}} \right\} + \lambda \sum_{i=n_1+1}^{m} \left\{ (t_i - \tau)^2 (W_i + 1) e^{\alpha\{\lambda(t_i-\tau)+\tau\}} \right\} \right], \quad (14)$$

$$\frac{\partial^2 l}{\partial \alpha \partial \beta} = k\left[\alpha^{-1} \sum_{i=1}^{n_1} t_i (W_i + 1) e^{\alpha t_i} - \alpha^{-2} \sum_{i=1}^{n_1} \{(W_i + 1)(e^{\alpha t_i} - 1)\} \right] - k\left[\alpha^{-1} \sum_{i=n_1+1}^{m} \{\lambda(t_i - \tau) + \tau\} (W_i + 1) e^{\alpha\{\lambda(t_i-\tau)+\tau\}} - \alpha^{-2} \sum_{i=n_1+1}^{m} \left\{ (W_i + 1)\left(e^{\alpha\{\lambda(t_i-\tau)+\tau\}} - 1\right) \right\} \right], \quad (15)$$

$$\frac{\partial^2 l}{\partial \alpha \partial \lambda} = \sum_{i=n_1+1}^{m} (t_i - \tau) - \beta k \left[\sum_{i=n_1+1}^{m} \{\lambda(t_i - \tau) + \tau\}(t_i - \tau)(W_i + 1) e^{\alpha\{\lambda(t_i-\tau)+\tau\}} + 2\alpha^{-1} \sum_{i=n_1+1}^{m} (t_i - \tau)(W_i + 1) e^{\alpha\{\lambda(t_i-\tau)+\tau\}} \right], \quad (16)$$

$$\frac{\partial^2 l}{\partial \beta \partial \lambda} = -k \sum_{i=n_1+1}^{m} \left[(W_i + 1)(t_i - \tau) e^{\alpha\{\lambda(t_i-\tau)+\tau\}} \right], \quad (17)$$

The Fisher information matrix $I(\alpha, \beta, \lambda)$ is obtained by calculating the expectation of minus of above equations (15), (16) and (17). It can be seen that $(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ is approximately bivariate normal with a mean (α, β, λ) and covariance matrix $I^{-1}(\alpha, \beta, \lambda)$under certain mild regularity conditions. Typically $I^{-1}(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ is used to estimate $I^{-1}(\alpha, \beta, \lambda)$. The following approximation is a simpler and equally valid procedure to use.

$(\hat{\alpha}, \hat{\beta}, \hat{\lambda}) = N\left((\alpha, \beta, \lambda), I_0^{-1}(\hat{\alpha}, \hat{\beta}, \hat{\lambda}) \right)$, given that;

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$$I_0^{-1}(\hat{\alpha},\hat{\beta},\hat{\lambda}) = \begin{bmatrix} -\frac{\partial^2 inL}{\partial\alpha^2} & -\frac{\partial^2 inL}{\partial\alpha\partial\beta} & -\frac{\partial^2 inL}{\partial\alpha\partial\lambda} \\ -\frac{\partial^2 inL}{\partial\alpha\partial\beta} & -\frac{\partial^2 inL}{\partial\beta^2} & -\frac{\partial^2 inL}{\partial\beta\partial\lambda} \\ -\frac{\partial^2 inL}{\partial\alpha\partial\lambda} & -\frac{\partial^2 inL}{\partial\beta\partial\lambda} & -\frac{\partial^2 inL}{\partial\lambda^2} \end{bmatrix}_{(\hat{\alpha},\hat{\beta},\hat{\lambda})}, \qquad (18)$$

To obtain approximate confidence intervals for parameters $\Theta = (\alpha,\beta,\lambda)$, a bivariate normal distribution with mean (α,β,λ) and covariance matrix $I^{-1}(\alpha,\beta,\lambda)$ can be used. Thus, the $100(1-\gamma)\%$ approximate confidence intervals for parameters $\Theta = (\alpha,\beta,\lambda)$ are

$$\hat{\alpha} \pm Z_{\gamma/2}\sqrt{M_{11}},\ \hat{\beta} \pm Z_{\gamma/2}\sqrt{M_{22}} \text{ and } \hat{\lambda} \pm Z_{\gamma/2}\sqrt{M_{33}},$$

Where, M_{11}, M_{22} and M_{33} are the principal diagonal elements of the above covariance matrix in equation (18) and $Z_{\gamma/2}$ is the percentile of the standard normal distribution with right-tail probability $\gamma/2$.

SIMULATION STUDIES

In the simulation studies, the biases and mean square errors (MSEs) of the maximum likelihood estimators (MLEs) for different values of parameters $\Theta = (\alpha,\beta,\lambda)$ and τ were investigated. Also computed are the asymptotic confidence intervals of 99% and 95% based on the asymptotic distribution of the MLEs (Tables 1 & 2).

Consideration is given to the following three progressive censorship schemes:

$$Scheme\ I: W_1 = n-m, W_2 = 0, \dots, W_m = 0$$

$$Scheme\ II: W_1 = 0, W_2 = 0, \dots, W_m = n-m$$

$$Scheme\ III: \begin{cases} W_{\frac{m+1}{2}} = n-m, W_j = 0\ \forall\, j \neq \frac{m+1}{2}, if\ m\ is\ odd \\ W_{\frac{m}{2}} = n-m, W_j = 0\ \forall\, j \neq \frac{m}{2}, \quad if\ m\ is\ even \end{cases}$$

The estimation steps are described as follows:

1. Consider different combinations of the values of n, m, k & τ and the set of parameters $\Theta = (\alpha,\beta,\lambda)$.
2. Create a random sample of size $n \times k$ from the GD and sort it according to the equation (6). One of the methods of obtaining the Gompertz random variable is the inverse CDF method. For example, if U is uniform distribution U (0, 1), then $t = \frac{log\left[-\frac{\alpha}{\beta}log(1-U)+1\right]}{\alpha}$ follows Gompertz distribution.

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3. For given n and m, use the model in equation (6) to generate progressively first-failure censored data. Hence, the following data set can be considered.
    a. $T^{W}_{1:m:n:k} < \cdots < T^{W}_{n_1:m:n:k} < \tau < T^{W}_{n_1+1:m:n:k} < \cdots < T^{W}_{m:m:n:k}$
    b. where $W = (W_1, W_2, \dots, W_m)$ and $\sum_{i=1}^{m} W_i = m - n$
4. To compute the MLEs of the model parameters using the progressive first failure censored data, the Newton Raphson method is used.
5. Replicate the steps 2−4, 1000 times.
6. The average values of biases and MSEs associated with the MLEs of the model parameters are computed. Then the asymptotic variances and approximate confidence bounds (95% and 99%) of the model parameters are estimated.
7. The above steps are repeated for different combinations of the values of n, m, k & τ.

TABLE 1
THE SIMULATION RESULTS FOR THE GOMPERTZ DISTRIBUTION UNDER PROGRESSIVE FIRST-FAILURE CENSORING USING PALT
$\boldsymbol{\alpha = 0.4\beta = 0.7\lambda = 1.2\tau = 2}$

k	n	m	Θ	Censoring Scheme I			Censoring Scheme II			Censoring Scheme III		
				Bias	MSE	95% CI	Bias	MSE	95% CI	Bias	MSE	95% CI
						99% CI			99% CI			99% CI
1	25	10	α	0.2211	0.0501	0.381-1.024	0.1391	0.0407	0.235-1.092	0.195	0.0428	0.339-1.061
						0.318-1.191			0.183-1.381			0.284-1.185
			β	0.5081	0.252	0.766-1.857	0.4301	0.2546	0.626-2.069	0.49	0.2477	0.731-1.887
						0.640-2.126			0.492-2.403			0.600-2.172
			λ	0.1233	0.0204	0.602-2.818	−0.0805	0.1916	0.385-3.201	−0.0822	0.0407	0.531-2.311
						0.471-3.562			0.273-4.438			0.428-2.903
	50	25	α	0.026	0.0233	0.138-1.176	−0.1171	0.0258	0.149-0.548	−0.0625	0.0256	0.131-0.863
						0.097-1.622			0.121-0.661			0.096-1.141
			β	0.1477	0.0865	0.298-2.258	0.3833	0.1567	0.740-1.545	0.0604	0.0471	0.309-1.901
						0.196-3.079			0.638-1.727			0.195-2.520
			λ	0.509	0.0255	0.561-2.739	0.2023	0.1212	0.838-2.305	0.2039	0.0484	0.968-2.098
						0.437-3.507			0.716-2.670			0.865-2.229
	75	35	α	0.0374	0.0178	0.080-0.774	−0.0984	0.0207	0.113-0.509	−0.0636	0.0154	0.014-0.677
						0.010-0.889			0.055-0.568			−0.066-0.778

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			β	0.1279	0.0402	0.188-1.437	0.3211	0.12	0.552-1.468	0.0506	0.0427	0.026-1.500
						0.011-1.638			0.411-1.609			−0.181-1.693
			λ	0.05	0.0184	0.455-2.020	0.1712	0.0388	0.727-2.086	0.2047	0.0443	0.955-1.813
						0.225-2.261			0.531-2.191			0.824-1.945
2	25	10	α	−0.2081	0.0409	0.049-0.706	−0.0568	0.0135	0.189-0.635	−0.1906	0.0444	0.186-0.255
						0.030-1.017			0.176-0.758			0.179-0.263
			β	0.2302	0.0905	0.269-3.044	0.305	0.0883	0.675-1.464	0.2244	0.061	0.477-1.725
						0.161-4.432			0.568-1.634			0.357-2.100
			λ	1.153	1.3207	1.061-5.238	0.0127	0.1288	0.578-2.426	1.2128	1.5502	1.130-5.113
						0.797-6.817			0.460-3.027			0.891-6.465
	50	25	α	−0.1511	0.0356	0.079-0.437	−0.0633	0.0136	0.244-0.449	−0.1047	0.0221	0.130-0.459
						0.026-0.500			0.215-0.477			0.082-0.508
			β	0.0286	0.0753	0.199-1.227	0.3014	0.1057	0.831-1.150	0.1443	0.0431	0.422-1.243
						0.052-1.384			0.774-1.207			0.297-1.379
			λ	0.4846	0.2377	0.890-2.448	0.2714	0.128	1.121-1.801	0.5308	0.2821	1.039-2.421
						0.650-2.688			1.018-1.914			0.796-2.625
	75	35	α	−0.1220	0.0242	0.145-0.430	−0.0580	0.0124	0.213-0.500	−0.1145	0.0211	0.171-0.419
						0.104-0.471			0.173-0.530			0.135-0.455
			β	0.022	0.0293	0.391-1.042	0.3075	0.0987	0.743-1.251	0.0495	0.014	0.460-1.018
						0.294-1.129			0.667-1.328			0.376-1.102
			λ	0.4199	0.1709	0.934-2.265	0.2517	0.1071	1.002-1.901	0.48	0.2309	0.985-2.354
						0.718-2.471			0.867-2.016			0.773-2.566

TABLE 2
THE SIMULATION RESULTS FOR THE GOMPERTZ DISTRIBUTION UNDER PROGRESSIVE FIRST-FAILURE CENSORING USING PALT
$\boldsymbol{\alpha = 0.4, \beta = 0.7, \lambda = 1.2, \& \tau = 3}$

				Censoring Scheme I			Censoring Scheme II			Censoring Scheme III		

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k	n	m	Θ	Bias	MSE	95% CI	Bias	MSE	95% CI	Bias	MSE	95% CI
						99% CI			99% CI			99% CI
			α	0.0628	0.023	0.232-0.852	−0.0778	0.0214	0.149-0.700	−0.0976	0.0275	0.129-0.709
						0.189-1.034			0.116-0.908			0.098-0.911
1	25	10	β	0.2228	0.1001	0.430-2.005	0.3034	0.1614	0.365-2.635	0.1773	0.0918	0.254-2.776
						0.296-2.408			0.243-3.572			0.122-3.944
			λ	0.2166	0.0867	0.686-2.819	0.5071	0.301	0.779-3.569	0.5331	0.3005	0.794-3.445
						0.549-3.504			0.622-4.509			0.665-4.286
	50	25	α	−0.0267	0.0228	0.158-0.883	0.0251	0.0119	0.288-0.587	0.0282	0.0027	0.246-0.708
						0.119-1.141			0.268-0.641			0.209-0.835
			β	0.0273	0.0284	0.279-1.842	0.3161	0.1222	0.915-1.103	0.2826	0.0903	0.622-1.539
						0.190-2.477			0.874-1.132			0.499-1.758
			λ	0.1133	0.0306	0.756-2.170	−0.3622	0.1363	0.664-1.100	0.2644	0.11	1.092-1.978
						0.650-2.552			0.625-1.175			1.010-2.103
	75	35	α	0.0332	0.0109	−0.203-1.070	−0.0124	0.0236	0.213-0.581	−0.0250	0.006	0.391-0.904
						−0.413-1.259			0.159-0.635			0.326-0.980
			β	0.0712	0.0154	−0.489-2.061	0.2082	0.0515	0.550-1.246	−0.0531	0.0176	1.682-2.637
						−0.898-2.440			0.443-1.352			1.555-2.804
			λ	−0.0215	0.0112	0.490-1.906	−0.1584	0.0502	0.544-1.558	0.062	0.0108	0.461-3.003
						0.285-2.111			0.388-1.714			0.077-3.407
2	25	10	α	0.0505	0.0123	0.255-0.753	0.0627	0.0059	0.271-0.732	0.077	0.0156	0.299-0.788
						0.196-0.903			0.244-0.815			0.250-0.852
			β	0.3577	0.131	0.684-1.601	0.3565	0.1419	0.725-1.508	0.3998	0.1698	0.789-1.512
						0.567-1.809			0.599-1.643			0.684-1.652
			λ	0.4359	0.1967	0.758-3.449	−0.3164	0.1539	0.453-1.754	0.2414	0.1099	0.757-2.727
						0.607-4.351			0.356-2.149			0.609-3.320

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	50	25	α	0.0177	0.0014	0.196-0.609	0.054	0.0041	0.256-0.651	0.0548	0.0121	0.197-0.661
						0.145-0.669			0.211-0.676			0.149-0.739
			β	0.2107	0.0509	0.466-1.333	0.3093	0.0907	0.653-1.345	0.2845	0.088	0.521-1.437
						0.356-1.463			0.494-1.464			0.382-1.556
			λ	0.2983	0.0989	0.842-2.147	−0.2182	0.0748	0.653-1.310	0.1456	0.0358	0.782-1.869
						0.640-2.369			0.556-1.406			0.627-2.043
	75	35	α	0.0071	0.0014	0.264-0.561	0.0308	0.003	0.301-0.538	0.0543	0.0087	0.296-0.551
						0.224-0.581			0.280-0.578			0.260-0.588
			β	0.1479	0.0208	0.566-1.109	0.2054	0.0504	0.697-1.083	0.1583	0.0352	0.598-1.087
						0.487-1.178			0.668-1.172			0.549-1.146
			λ	0.2306	0.0603	0.861-2.086	−0.1596	0.043	0.732-1.368	0.1023	0.0192	0.828-1.766
						0.666-2.172			0.649-1.491			0.665-1.901

CONCLUSION

In this research, the classical inference approach for the unknown parameters of the Gompertz distribution (α, β) and the acceleration factor (λ) is addressed when data from step-stress partially accelerated life tests is censored for progressive-first-failure. Due to the fact that maximum likelihood estimators cannot be produced in closed form, the computation has been performed iteratively using Newton Raphson. Additionally, the model parameters' approximate confidence intervals are produced. Calculations are performed using a variety of sample sizes $(n \times k)$, a variety of stress change points (τ), and three distinct progressive censoring strategies (I, II, III). Monte Carlo simulations are used to evaluate the estimators' performance, which is shown to be pretty adequate. The findings indicate that as the sample size increases, the MSEs of the three estimators α, β, and λ decrease. It can be seen that α has a lower MSE than both β and λ. Additionally, we observe a decrease in the MSEs for α as τ increases. For k = 1, on the other hand, the MSEs for λ increase as τ increases, while the MSEs for β decrease. However, when k = 2, the MSEs for λ decrease as τ increases, while the MSEs for β increase. It's difficult to determine which censoring scheme is the most effective. By comparing the MSEs of the estimators α, β, and λ for each censoring scheme in Table 1, it is possible to conclude that for k = 1, the best censoring scheme for both α and λ is III, and for β, it is I. For k = 2, the optimal censoring scheme is II for both α and λ, and III for l. As shown in Table 2, the optimal censoring scheme for β and λ is I, while the optimal censoring scheme for α is III. For k = 2, the optimal censoring scheme for α and β is I, while the optimal censoring scheme for λ is III. When it comes to the interval estimation, the second scheme (II), in which censoring occurs after the

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most recent observed failures, mostly produces shorter lengths than the other two schemes. The possible exception could be due to a data fluctuation.

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