POINT ELASTICITY VERSUS ARC ELASTICITY: ON DIFFERENT APPROACHES TO TEACHING ELASTICITY IN PRINCIPLES COURSES

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ABSTRACT

While most principles texts seemingly follow a similar routine when explaining the concept of elasticity, a closer look at the explanations shows that there is plenty of variety in the approaches taken by different authors. What mostly remains hidden in those explanations is their reliance on point elasticity, arc elasticities, or the mixture of both. We provide an overview of point elasticity and arc elasticity, and assess different approaches that can be found in contemporary principles texts in terms of their consistency with these two concepts.

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THE TWO PUZZLES

This work is motivated by two puzzles that one might notice in some principles texts when it comes to explaining the concept of elasticity. The first puzzle is the unconventional treatment of the notion of the percentage change in a variable. One would think that there is one and only one proper way to calculate percent change:

$$Percent\ Change\ in\ X = \frac{X_{new} - X_{old}}{X_{old}} \times 100\%. \qquad (1)$$

However, as soon as it is declared that price elasticity of demand compares the percentage change in quantity demanded with the percentage change in price and is calculated as the following ratio,

$$Elasticity\ of\ Demand = -\frac{Percent\ Change\ in\ Q}{Percent\ Change\ in\ P}, \qquad (2)$$

where Q is quantity demanded, and P is price, most principles texts express dissatisfaction with the lack of symmetry in the results provided by (1) and come up with the following replacement that is commonly referred to as the midpoint method:

$$Percent\ Change\ in\ X = \frac{X_{new} - X_{old}}{(X_{new} + X_{old})/2} \times 100\% \qquad (3)$$

(Bade and Parkin 2015; Chiang 2014; Colander 2013; Krugman and Wells 2014; Samuelson and Nordhaus 2015; Schiller and Gebhardt 2016).

In Krugman and Wells' words, "the midpoint method replaces the usual definition of the percent change in a variable, X, with a slightly different definition" (Krugman and Wells 2014). For someone who is sensitive to the rigor in definitions, this practice might represent an uncomfortable puzzle. No matter how "slight" the difference between the usual definition of the percent change and the midpoint method is, (1) and (3) are different formulas, and one might argue that referring to both, while using exactly the same term, "percent change", is inconsistent: if percentage change is (1) then it cannot be (3), if percentage change is (3) it cannot be (1).

The second puzzle is how the midpoint method based definition of elasticity,

$$E = -\frac{Q_1 - Q_0}{P_1 - P_0}\frac{\bar{P}}{\bar{Q}}, \qquad (4)$$

where $\bar{Q}$ and $\bar{P}$ are the averages of Q_0 and Q_1, and P_0 and P_1 respectively, is used by some textbook authors to estimate elasticity at a point on the linear demand curve (Bade and Parkin 2015). The problem that one might notice is that when applied to estimate elasticity at a point on the linear demand curve, (4) provides results that are identical to those delivered by the formula that is based on the standard definition of percentage change and that effectively gets dismissed:

$$E = -\frac{Q_1 - Q_0}{P_1 - P_0}\frac{P_0}{Q_0}. \qquad (5)$$

To notice this equivalency, we need to recall the Marshallian definition of point elasticity,

$$E(p,q) = -\frac{dq}{dp}\frac{p}{q}, \qquad (6)$$

and then compare it to (4) and (5). It is easy to see that when dealing with the linear demand curve, (4), (5), and (6) provide identical estimates of elasticity at a point as long as the point of interest is identified as the midpoint ($\bar{Q}$, $\bar{P}$) in (4), the starting point (Q_0, P_0) in (5), and (q, p) in (6).

This observation makes the first puzzle about the modification of the standard percentage change formula even more bewildering: why bother with this modification if in the end the resulting formula provides exactly the same estimates that can be provided by a measure that requires no modification, and, therefore, is more straightforward?

As we show below, the second puzzle can be resolved if one realizes that there is not just one, but two notions of elasticity that many principles text authors implicitly discuss without even mentioning them: point elasticity and arc elasticity. Thus, before we proceed, we provide a quick overview of the two concepts, their origins, and development.

POINT ELASTICITY AND ARC ELASTICITY: UNDERSTANDING THE DIFFERENCE

While point elasticity (6) is a straightforward and well-known concept that has not changed since it was introduced by Marshall (1890), arc elasticity is a fairly evasive and obscure one that has gone through a number of revisions since it was introduced by Dalton (1920). Thus, most of the discussion in this section is about arc elasticity.

Arc elasticity came into existence under rather peculiar circumstances due to an inconsistency in Marshall's discussion of his calculus-based notion of elasticity that is equivalent to (6), and that later was called point elasticity by Dalton (1920). The inconsistency, which has been noticed by Dalton (1920), is between what we will call Marshall's one percent change in price scenario and his calculus-based definition of point elasticity.

Here is Marshall's (1890) one percent change in price scenario that he opens his formal discussion of elasticity with: "…we may say that the elasticity of demand is one, if a fall of one percent in price will make an increase of one percent in the amount demanded; that is two or a half, if a fall of one percent in price makes an increase of two or one half percent respectively in the amount demanded; and so on."

This is not that there is something wrong with this scenario, if one wants to use it as a definition of elasticity. The problem is that, unless we are measuring elasticity at a point on the linear demand curve, it is inconsistent with the Marshall's (1890) definition of elasticity that immediately follows in the text and which is equivalent to the calculus based definition of elasticity (6). Dalton (1920) notices this inconsistency and convincingly argues that there is an error.

Interestingly enough, instead of identifying the one percent change in price scenario as an approximation of point elasticity that is based on infinitesimal changes in price, and which Marshall clearly had in mind, Dalton (1920) decided to give the one percent change in price scenario a life of its own under the name of arc elasticity, arguing that "[e]lasticity at a point, a conception derived from the differential calculus, is only the elasticity corresponding to infinitesimal changes in demand price and amount demanded… The distinction, on which I have here insisted, is between elasticity at a point and elasticity across a finite arc, or, as we may say more shortly, between *point elasticity* and *arc elasticity*." To be clear, Dalton (1920) did not introduce his own definition of arc elasticity, but simply interpreted Marshall's one percent change in price scenario as such.

This is how arc elasticity came to life, thus, for the most part owing its existence to Marshall's ambiguity regarding the meaning of the one percent change in price scenario in his discussion of the calculus based concept of point elasticity. One might even wonder if it was a worthy cause to use the finite change in price scenario as a basis for a separate definition of a calculus based concept, rather than simply interpreting it is as an approximation of the calculus based one. Nevertheless, rather than getting dismissed or ignored, arc elasticity attracted some interest from a number of scholars: Schultz (1928), Lerner (1933), and Allen (1934) came up with measures of their own to improve Dalton's definition of arc elasticity.

We are focusing on Allen's measure of arc elasticity, as clearly this is what the midpoint method based formula (4) replicates, even if neither Allen's name, nor the term arc elasticity itself are mentioned, and even if sometimes it gets misinterpreted.

Allen (1934) proposed to take into account the following three important considerations when choosing a measure of arc elasticity:

1. It is essential that the formula for arc elasticity should be independent of the units of measurement of x and p.
2. It is desirable that the formula should be symmetrical with respect to P_1 and P_2 and not dependent on the taking of one end of the arc as a "base."
3. It is desirable that the formula should be give a value unity whenever the outlays in the situations represented by the points P_1 and P_2 are equal, i.e. whenever $p_1x_1 = p_2x_2$.

Eventually, Allen (1934) comes up with (4) as a measure of arc elasticity that satisfied all three requirements listed above, and, therefore, is a worthy replacement of Dalton's measure of arc elasticity that does not satisfy requirements 2 and 3.

POINT ELASTICITY AND ARC ELASTICITY IN PRINCIPLES TEXTS: RESOLVING THE TWO PUZZLES

Based on the information presented in the previous section, we provide an explanation to each of the two puzzles listed in the beginning of this work. The first puzzle is about the unconventional use of the term "percentage change" that is assigned not to the standard percentage change formula (1), but to the midpoint method (3). It is clear why one would want to use (3) instead of (1), as this is the way to come up with (4) as a measure of arc elasticity. However, what is not clear is what is the point of sticking to the notion of percentage changes, if the standard way to calculate it does not satisfy one's requirements? Would not it be easier to define arc elasticity as a ratio of two relative changes or even as a ratio of two midpoint changes, because this is what eventually is being used?

One could justify this practice as a way to preserve consistency between Marshall's "canonical" definition of arc elasticity, if this is how we interpret his one percent change in price scenario, and Allen's measure of arc elasticity. Dalton's definition of arc elasticity was nothing more than a specific interpretation of Marshall's one percent change in price scenario as a measure of arc elasticity, which means that if we want to switch from Dalton's to Allen's measure of arc elasticity, then we cannot use Marshall's one percent change in price scenario as a basic definition of arc elasticity either, not as long as we believe that percentage changes are calculated as (1). Thus, instead of revising Marshall's one percent change in price based scenario—something that Allen (1934) effectively did—most principles text authors prefer to loosen the definition of percentage change, implying that one can modify the standard formula for calculating percentage changes and still call its results percentage changes.

Now, we can also address the second puzzle to figure out why some principles texts, like Bade and Parkin (2015), would use (4) to estimate elasticity at a point on the linear demand curve, instead of applying (5) that would provide exactly the same results, as long as those are properly interpreted. Clearly, the authors use the midpoint method formula (4) for two purposes: one, to estimate arc elasticity between two points on the demand curve, and, another, to estimate point elasticity at the midpoint on the linear demand curve. Technically, as we explained above, (4) does provide an accurate estimate of point elasticity at the midpoint on the linear demand curve. However, as we argue below, mixing arc elasticity and point elasticity together is likely to result in inconsistency and confusion, and is not a good idea in principles courses.

DIFFERENT APPROACHES TO EXPLAINING ELASTICITY: THEIR ADVANTAGES AND DISADVANTAGES

Bade and Parkin (2015) applying (4) to measure point elasticity on the linear demand curve after introducing the formula as a measure of arc elasticity is probably the most conspicuous example of mixing arc elasticity and point elasticity in our sample of ten contemporary textbooks. Overall, six out of ten principles texts in our sample to a certain extent mix these two concepts by introducing (4) as a measure of arc elasticity between two points, and then referring to elasticity measured at a point on the demand curve, that is, point elasticity. We consider this practice as the first approach out of four that we describe in this section, and it can be found in the following principles texts: Bade and Parkin (2015), Chiang (2014), Colander (2013), Krugman and Wells (2014), Samuelson and Nordhaus (2015), and Schiller and Gebhardt (2016).

To see how principles texts' authors that follow this approach make a transition from arc elasticity to point elasticity one can take a look in Krugman and Wells (2014). After convincingly arguing that (4) provides the desired results when estimating elasticity between two points on the demand curve, they end up with the following "clarification" later on in the text:

"Suppose an economist says that "the price elasticity of demand for coffee is 0.25." What he or she means is that at the current price the elasticity is 0.25. In the previous discussion of the tall bridge, what we were really describing was the elasticity at the price of $0.90. Why this qualification? Because for the vast majority of demand curves, the price elasticity of demand at one point along the curve is different from the price elasticity of demand at other points along the same curve."

One might argue that this passage is inconsistent, given that the authors just discussed an example where they are measuring elasticity between two points on the demand curve as price goes up from $0.90 to $1.10, thus, referring to arc elasticity, and now they are switching to estimating elasticity at the starting point of this change, thus, referring to point elasticity.

As we explained above, arc elasticity is a concept based on finite changes in quantity demanded and price between two points on the demand curve. Point elasticity is a concept based on infinitesimal changes in quantity demanded and price from the point on the demand curve. As the discussions switches from estimating elasticity between two points on the demand curve to a single point, one should wonder: what has happened to those finite changes in price and quantity demanded that were discussed in the beginning? The proper answer, of course, is that those changes got infinitesimal and the two points merged. Without making this clear, it is impossible to consistently switch from arc elasticity to point elasticity. We argue that by making this switch without proper explanation, many principles text authors are confusing the students and instructors who are using their texts, and quite often show the lack of clarity in understanding the distinction between arc elasticity and point elasticity themselves.

The practice of switching from finite changes in price and quantity to infinitesimal ones is likely to be even more confusing to people with advanced degrees in economics as they are already familiar with the concept of point elasticity and would recognize point elasticity in the discussion of elasticity at a point on the demand curve. If at the same time they are not familiar with the concept of arc elasticity, they might feel confused about all the effort that the followers of the first approach are taking in the beginning to come up with (4), if they end up estimating elasticity at a point.

That was exactly one of the puzzles that motivated this work, as we observed Bade and Parkin (2015) using the midpoint formula to measure elasticity at a point on the linear demand curve. Frank and Bernanke's (2009) criticism of the midpoint formula appears to be related to this type of confusion, as they insist that the lack of symmetry in (5) that is commonly presented in principles texts when introducing the midpoint formula does not justify an introduction of the midpoint method, but can be resolved by realizing that point elasticity varies as one moves along the linear demand curve. Frank and Bernanke (2009) argues that "To have elicited a uniquely correct answer, [the question] should have been "What is the price elasticity of demand at point A" or "What is the price elasticity at point B?" They go on, declaring that "We will not employ the midpoint formula again in this text. Hereafter, all questions concerning elasticity will employ the measure … called *point elasticity*."

Being true to his word, Frank & Wells (2009) (the second approach) rely on point elasticity only, while measuring it on the linear demand curve by applying (5). The biggest advantage of this approach, compared to the first one, is that Frank and Bernanke (2009) avoids inconsistency as they are not mixing point elasticity and arc elasticity together. Also, this approach is more streamlined as it completely skips on the first part of the first approach where the standard percentage change formula is modified and the midpoint formula (4) is introduced.

However, there are two problems with the second approach. First, formally speaking, this approach is not consistent with the notion of point elasticity that is supposed to be based on infinitesimal changes in price and quantity. It is a convenient coincidence that (5) provides accurate estimates of point elasticity at the starting point on the linear demand curve when a move between two points on the curve is considered, but (5) is not a proper measure of point elasticity, unless this move gets infinitesimally small.

Another problem with this approach is related to the relationship between elasticity and changes in price when demand is elastic or inelastic. When the calculus based definition of point elasticity, (6), or Allen's measure of arc elasticity, (4), are used, the following is always true:

1. When price elasticity of demand is greater than 1, changes in total revenue in response to changes in price and changes in total revenue always move in opposite directions.

2. When price elasticity of demand is smaller than 1, changes in price and changes in total revenue always move in the same direction.
3. When price elasticity of demand is 1, changes in price do not affect total revenue.

However, 3 is never true, and 1 and 2 may not be true when (5) is applied to measure elasticity at a point on the linear demand curve. 3 is never true because to get an estimate of price elasticity equal to 1 on the linear demand curve, one should pick the starting point exactly where total expenditure is at its maximum. Thus, while moving away from that point on the demand curve we will always get a decrease in total revenue. 1 and 2 are only true if both the starting and the ending points on the demand curve are located in the area of elastic or inelastic demand. If the ending point crosses into the other area and the change in price is large enough, either statement can be violated. Thus, for 1 and 2 to be true when using this approach, one should keep changes in price sufficiently small and/or stay sufficiently far away from the point on the demand curve where total revenue is maximized.

Just like Frank and Bernanke (2009), Stiglitz & Walsh (2006) (the third approach) use (5) when estimating elasticity, however, unlike Frank and Bernanke , who estimate elasticity at a point, Stiglitz & Walsh measure elasticity between two points, that is, arc elasticity. This is a rather peculiar approach, as Stiglitz & Walsh (2006) are applying Dalton's measure of arc elasticity while solving the lack of symmetry in (5) by considering upward movements in price only. Accordingly, Stiglitz and Walsh (2006) provide the following definition: "Price elasticity of demand: the percentage change in the quantity of a good demanded as a result of a 1 percent increase (change) in the price charged." While "change" is mentioned in the definition in parentheses, implying that downward movements in price could be considered as well, Stiglitz and Walsh (2006) effectively ignore it as in their numerical examples related to measuring elasticity on the linear demand curve they only consider upward movements in price.

Such a lack of symmetry is one disadvantage of this approach, and it is quite surprising that Stiglitz & Walsh (2006) decided not to use Allen's measure of arc elasticity that is free from this problem. Also, Stiglitz & Walsh's approach suffers from the same disadvantage that Frank's approach does when it comes to connecting elasticity and changes in price and total revenue on the linear demand curve that we described above: that is, 3 is never true for this approach, and 1 and 2 may not be true either.

Finally, the fourth approach that is followed by Mateer & Coppock (2014) & McConnell et al. (2015) in our sample, is the most consistent of all, in our opinion. Just like the first one, this approach starts with introducing the midpoint formula and applying it to measure arc elasticity between two points on the demand curve. However, unlike the first approach, this one never slips into considering elasticity at a point, even when the linear demand curve is considered, thus, consistently sticking to arc elasticity all the way through.

CONCLUSION

In this work, we set a goal to resolve two puzzles that one might notice in some principles texts. One puzzle is the replacement of the standard percentage change formula with the midpoint method, while still referring to that measure as a percentage change formula. We conclude that this practice allows to preserve consistency between the Marshall's one percent change in price scenario, that has been interpreted as a definition of arc elasticity by Dalton (1920), & Allen's (1934) measure of arc elasticity that effectively replaces the one percent change in price scenario with the midpoint change based definition of arc elasticity.

For someone who is not concerned with preserving this consistency, but feels strongly that the standard formula for calculating percentage changes (1) is the only way to do so, we could suggest to avoid defining elasticity as a ratio of two percentage changes and from the very beginning define it as a ratio of two midpoint (percentage) changes. That probably would be helpful to the students as well as they would be less likely to get an impression that economists in general calculate percentage changes in a way that differs from the conventional formula.

Another puzzle is the use of the midpoint based formula (4) to estimate elasticity at a point on the linear demand curve by some books, instead of using the standard percentage change based formula (5) that provides exactly the same result on such a curve. We conclude that in such a case the midpoint based formula (4) is used for two purposes: to measure arc elasticity between two points and to measure point elasticity at the midpoint on the demand curve. Thus, in such a case arc elasticity and point elasticity are being implicitly mixed together. Despite the majority of principles texts in our sample (six out of ten) follow the above approach of mixing arc elasticity and point elasticity, we argue that mixing these two concepts should be avoided in principles courses.

We conclude that out of four approaches that we identified in our sample of ten principles texts, the most robust and consistent one is to use Allen's formula exclusively to measure elasticity between two points on the demand curve while avoiding any references to elasticity at a point on the demand curve.

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